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Dominance of Milnor attractors in globally coupled dynamical systems with more than 7 ± 2 degrees of freedom

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The prevalence of Milnor attractors has recently been reported in a class of high-dimensional dynamical systems. We study how this prevalence depends on the number of degrees of freedom by using a globally coupled map and show that the basin fraction of Milnor attractors increases drastically around 5–10 degrees of freedom, saturating for higher numbers of degrees of freedom. It is argued that this dominance of Milnor attractors in the basin arises from a combinatorial explosion of the basin boundaries. In addition, the dominance is also found in a system without permutation symmetry, i.e., a coupled dynamical system of nonidentical elements. Possible relevance to the magic number 7 ± 2 in psychology is briefly discussed.

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In recent studies of dynamical systems with many degrees of freedom, the prevalence of Milnor attractors has been recognized [1,2]. The Milnor attractors are defined as follows: an arbitrary small perturbation to an orbit at a Milnor attractor can kick the orbit away from it to a different attractor, even though a finite measure of initial conditions is attracted to the attractor by temporal evolution [3–5]. In other words, the basin of the attractor touches the attractor itself somewhere. An orbit is often attracted to a Milnor attractor, but can be kicked away from it by infinitesimal perturbation.

It should be noted that Milnor attractors can exist in low-dimensional dynamical systems such as a two-dimensional map as well. When changing the parameter of a dynamical system, the basin boundary of an attractor may move until, for a specific value of the parameter, the basin boundary touches the attractor. If, for this parameter value, the attractor has a positive measure of initial conditions forming the basin of attraction, it becomes a Milnor attractor [3]. It is naively expected that this situation occurs only for very specific parameter values (e.g., at a bifurcation point), and that the Milnor attractors may not exist for an open set of parameter values.

In recent studies of globally coupled dynamical systems, however, Milnor attractors are found to be sometime prevalent, occurring not only for specific isolated parameter values, but for continuous ranges of parameter values. Furthermore, the measure of the initial conditions that belong to the basin of these Milnor attractors is a relatively large proportion of the phase space. Indeed, for some parameter ranges, almost all randomly chosen initial conditions fall onto Milnor attractors [1,2].

Such dominance of Milnor attractors is often found, for example, in globally coupled maps (GCM) [6] with 10 degrees of freedom or so [7], and within a range of parameter values where many attractors coexist. The question we address in the present paper is why can there be so many Milnor attractors in a ‘‘high-dimensional’’ dynamical system, and how many elements are sufficient for constituting such ‘‘high’’ dimensionality. With the help of numerical results obtained from simulations of globally coupled maps, we will show that the dominance starts to be common at 5–10 degrees of freedom. We propose a possible origin for this dominance of Milnor attractors, by noting a combinatorial explosion of basin boundaries. We will also show that the prevalence of Milnor attractors is observed even in a system without symmetry, i.e., in a coupled dynamical system of nonidentical elements. Finally we briefly discuss the possible relevance of our results to the magic number 7 ± 2 in psychology.

As a prototype example to study this problem we use a GCM [6]

$$x_{n+1}(i)=(1-\epsilon)f(x_n(i))+\frac{\epsilon}{N}\sum_j f(x_n(j)), \qquad (1)$$

where n is the discrete time and i being the index for its elements ($i=1,2,\ldots,N=$ dimension of the system). For the elements we choose $f(x)=1-ax^2$, since the model has been thoroughly investigated as a prototype model for high-dimensional dynamical systems. The coupling parameter ϵ is fixed at 0.1, since for this value the typical behaviors of the above GCM that are relevant here can be observed by changing only a.

In the present model, each attractor can be coded by the so-called clustering condition, that is to say, by the way how the N elements of the system partition into mutually synchronized clusters, i.e., a set of elements in which $x_n(i)=x_n(j)$ [6]. Each attractor is coded by the number of clusters k and the number of elements N_k in each cluster with the clustering condition given by $(N_1,N_2,\ldots,N_k)$. Indeed, for a GCM with $N=10$ [1], many initial conditions are attracted to a Milnor attractor. Here, in order to discuss the dependence of the size of the basin fraction of the Milnor attractors on the number of degrees of freedom, we have computed the ratio of initial conditions that are attracted to Milnor attractors.

As shown in Fig. 1, the basin fraction of Milnor attractors is large around $a\approx1.65$, especially for $N\geqslant5$. Its parameter dependence, however, is quite strongly dependent on N, at least when N is not so large. Hence, it is not that relevant to compare the behaviors for different N at a given value of a.

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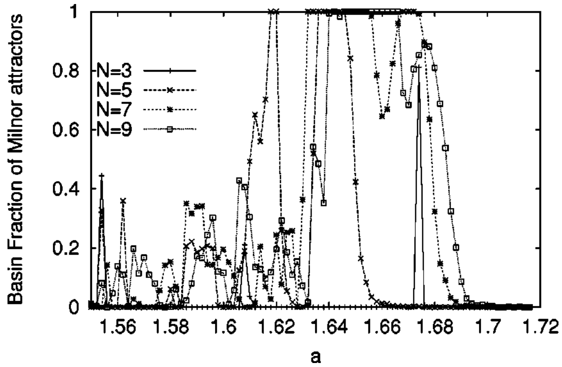

FIG. 1. The basin fraction of Milnor attractors plotted as a function of the parameter a, for $N=3, 5, 7,$ and 9. For the present simulations, we take 1000 randomly chosen initial conditions, and iterate 10^5 steps. Then the orbit is perturbed as $x_n(i)+10^{-10}\sigma_i$, with σ_i as a random number over $[-0.5,0.5]$. With 10^3 trials of such perturbations, we checked whether the orbit remains on the same attractor or not, after 5×10^4 time steps. If some of the 10^3 trials result in an escape from the original attractor, it is regarded as a Milnor attractor [8].

Instead, we compute the average basin fraction of Milnor attractors over the parameter interval $1.55<a<1.72$. In Fig. 2, this average is plotted as a function of the number of degrees of freedom N. The increase of the average basin fraction of Milnor attractors with N is clearly visible for $N \approx (5-10)$, while it levels off for $N>10$.

Now we discuss how the dominance of Milnor attractors appears. In a system with identical elements, due to the symmetry, there are at least

$$M(N_1,\ldots,N_k)=(N!/\Pi_{i=1}^{k}N_i!)\Pi_{\text{oversets of } N_i=N_j}(1/m_\ell!)$$

attractors for each clustering condition, where m_ℓ is the number of clusters with the same value N_j [1]. Then, combinatorial increase in the number of attractors can be expected when many of the clustering conditions are allowed as attrac-

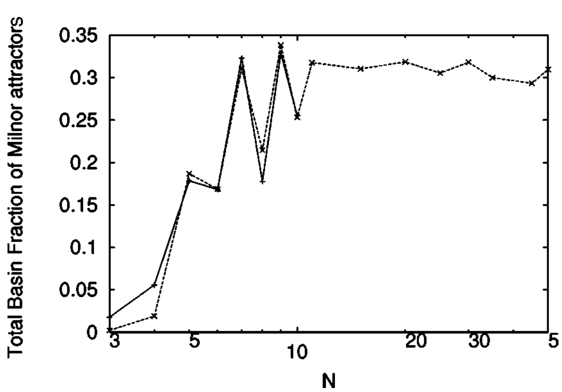

FIG. 2. The average fraction of the basin ratio of Milnor attractors. After the basin fraction of Milnor attractor is computed as in Fig. 1, the average of the ratios for parameter values $a=1.550,1.552,1.554,\ldots,1.72$ is taken. This average fraction is plotted as a function of N. (The dotted line depicts the case where a is incremented by 0.01 instead of 0.002; for $N>10$, the simulation is carried out only with this choice.)

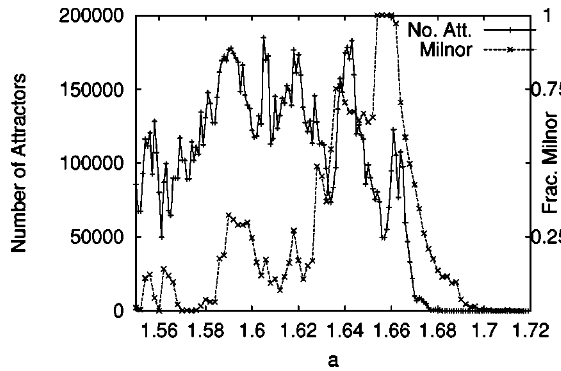

FIG. 3. The number of attractors $(+)$ estimated from simulations over 10^5 initial conditions. The estimated number of attractors is plotted as a function of a. $N=10$. All attractors that are concluded to exist by the symmetry argument are also counted. The basin fraction of Milnor attractors obtained in the same way as in Fig. 1 is also plotted by a dotted line with $\times$.

tors. However, the increase of the number of attractors cannot explain the increase of the basin for Milnor attractors. In Fig. 3. we have plotted the number of attractors and the basin fraction of Milnor attractors for $N=10$. As can clearly be seen, the dominance of the Milnor attractors is not necessarily observed when the number of attractors is high. Rather, the fraction of Milnor attractors gets large even when many attractors start to disappear with the increase of a.

For the parameter region where many attractors start to disappear, there remain basin boundary points separating such (collapsed) attractors and the remaining attractors. To explain the prevalence of Milnor attractors, we discuss how the distance between an attractor and its basin boundary changes with N. Consider a one-dimensional phase space, and a basin boundary that separates the regions of $x(1)>x^*$ and $x(1)<x^*$, while the attractor in concern exists at around $x(1)=x_A<x^*$, and the neighboring one at around $x(1)=x_B>x^*$. Now consider a region of N-dimensional phase space $x_A<x(i)<x_B$. If the region is partitioned by (basin) boundaries at $x(i)=x^*$ for $i=1,\ldots N$, it is partitioned into 2^N units. Since this partition is just a direct product of the original partition by $x(1)=x^*$, the distance between each attractor and the basin boundary does not change with N. [For example, consider the extreme case that N identical maps are uncoupled $(\epsilon=0)$.]

On the other hand, consider a boundary given by some condition for $[x(1),\ldots,x(N)]$. In the present system with global (all-to-all) couplings, many of permutational change of $x(i)$ in the condition give also basin boundaries. Here the condition for the basin can also have clustering $(N_1,\ldots,N_k)$, since the attractors are clustered as such. Then there are $M(N_1,\ldots,N_k)$ partitions by boundaries equivalent by permutations. The number of regions partitioned by the boundaries increases combinatorially with N. Roughly speaking, it increases in the order of $(N-1)!$, when the boundary has a variety of clusterings (i.e., large M). Now the N-dimensional phase space region is partitioned by $O(N-1)!)$ basin boundaries. Recalling that the distance between an attractor and the basin boundary remains at the same order for the partition of the order of 2^N, the distance should decrease if $(N-1)!$ is larger than 2^N. Since for $N>5$, the former increases drastically faster than the latter, the distance should decrease drastically for $N>5$. Then for $N>5$, the probability that a basin boundary touches with an attractor itself will be increased. Since this argument is applied for any attractors and their basin boundary characterized by complex clusterings having combinatorially large $M(N_1, \ldots, N_k)$, the probability that an attractor touches its basin boundary is drastically amplified for $N>5$. In fact, for a certain parameter regime ($1.64<a<1.68$ in the present case), basin boundaries with such partitions are dominant, and the ratio of Milnor attractors is increased. Although this explanation may be rather rough, it gives a hint to why Milnor attractors are so dominant for $N \gtrsim (5\text{–}10)$.

Since the above discussion is based mainly on simple combinatorial arguments, what we need is instability in orbits leading to many attractors, and global (all-to-all) coupling to allow for permutations of elements. Then, the dominance of Milnor attractors for $N \gtrsim (5\text{–}10)$ may be rather common in globally coupled dynamical systems. To check this possibility, we have also made numerical simulations for Josephson junction series arrays that are globally coupled through a resistive shunting load and driven by an rf bias current [9], given by $\ddot{\phi}_j + g\dot{\phi}_j + \sin\phi_j + g\sigma/N\Sigma_{m=1}^{N}\dot{\phi}_m = i_{dc} + i_{rf}\sin(\Omega_r f t)$. By using the parameter values adopted in Ref. [9], we have computed the basin volumes for Milnor attractors, at the partially ordered phase where a variety of attractors with many clusters coexists. Again, the basin volumes are close to 0 for $N \leqslant 4$, and increase at $5<N<10$ [12]. It is also interesting that pulse-coupled oscillators with global coupling also show the prevalence of Milnor attractors for $N \geqslant 5$ [2].

So far reports on the prevalence of Milnor attractors have been limited to a system with symmetry. For example, in the GCM (1), the permutation symmetry arising from employing identical elements leads to a combinatorial explosion in the number of attractors as mentioned. Then, one may wonder whether the prevalence of Milnor attractors is possible only for such highly symmetric systems, especially because some Milnor attractors are known to disappear when introducing tiny asymmetries [10]. We have therefore studied a GCM with inhomogeneous parameters [11], given by $x_{n+1}(i) = (1-\epsilon)f_i(x_n(i)) + (\epsilon/N)\Sigma_j f_j(x_n(j))$ with $f_i(x) = 1 - a_i x^2$, and $a_i = a_0 + a_w(i-1)/(N-1)$.

In Fig. 4, we have plotted the basin fraction for Milnor attractors with the change of the parameter a_0 while fixing $a_w = 0.1$, for $N = 5\text{–}12$. Although the fraction is smaller than in the homogeneous case, Milnor attractors are again observed and their basin volume is rather large for some parameter region. As in the symmetric case, the basin fraction of Milnor attractors increases around $N \approx (5\text{–}10)$. (The fraction is almost zero for $N \leqslant 4$). Note that eventhough complete synchronization between two elements is lost, clusterings as with regards to the phase relationships can exist. For example, there are two groups when considering the oscillations of phases as large-small-large . . . and small-large-small . . . , that are preserved in time for many attractors. Similarly, it is natural to expect an explosion in the number of the basin boundaries for some parameter regime. Accordingly the argument on the dominance of Milnor attractors for a homogeneous GCM can be applied here to some degree as well [13].

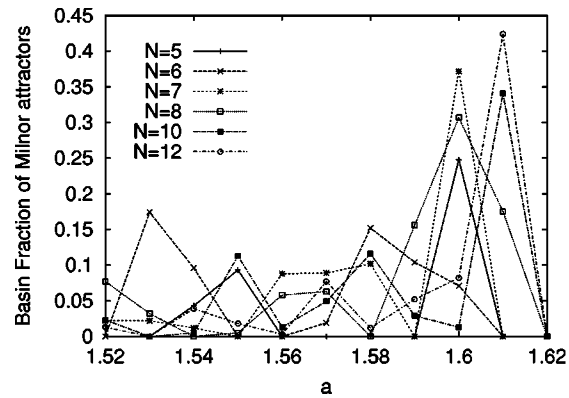

FIG. 4. The basin fraction of Milnor attractors for a GCM with inhomogeneous parameters. Here $a(i) = a_0 + 0.1 \times (i-1)/(N-1)$. Plotted as a function of the parameter a_0, for $N = 5-8$, 10, and 12. (For $N \leqslant 4$, the fraction is almost zero for all a_0.) The fraction is computed in the same way as in Fig. 1, except that 100 trials of perturbations were used instead of 1000. Since the clustering condition cannot be used in this case, we checked whether the orbit is on the same attractor, by computing the temporal average of $x_n(i)$ over 5×10^6 steps, before and after each perturbation. If the average agrees to within a precision of 10^{-3}, the orbit is regarded to be on the same attractor.

The term magic number 7 ± 2 was originally coined in psychology [14], when it was found that the number of chunks (items) that is memorized in short term memory is limited to 7 ± 2 (see Ref. [15] for possible relationship with chaos). To memorize k chunks of information including their order (e.g., a phone number of k digits) within a dynamical system, let us assign each memorized state to an attractor of a k-dimensional dynamical system, as is generally adopted in neural network studies. In this k-dimensional phase space, a combinatorial variety of attractors has to be presumed in order to assure a sufficient variety of memories. Depending on the initial condition (given by inputs), an orbit has to be separated to different attractors. Then, a combinatorial explosion of basin boundaries is generally expected with the increase of k, if the neural dynamics in concern are globally coupled (as often adopted in neural networks). Then, following the argument in the present paper, Milnor attractors may be dominant for $k>(5\text{–}10)$. (Recall that the number does not strongly depend on the choice of models, since it is given by the combinatorial argument.) Since the state represented by a Milnor attractor is kicked out by tiny perturbations, robust memory may not be possible [16] for information that contains more than 7 ± 2 chunks [19]. Although this explanation is a rough sketch, it can possibly be applied to other systems that adopt attractors as memory.

In dynamical systems, it is well known that the dimensional cutoff $\geqslant 3$ plays an important role for the existence of chaos. It is interesting then to investigate whether there are certain higher dimensions that similarly form dimensional boundaries beyond which the behavior of a dynamical system changes qualitatively. The present study may shed new light on this possibility. Also, it is interesting to note that in Hamiltonian dynamics, agreement with thermodynamic behavior is often observed only for degrees of freedom higher than 5–10 [20]. Considering the combinatorial complexity woven by all the possible Arnold webs (that hence may be termed "Arnold spaghetti"), the entire phase space volume that expands only exponentially with the number of degrees of freedom may be covered by webs, resulting in uniformly chaotic behavior. If this argument holds, the degrees of freedom required for thermodynamic behavior can also be discussed along the line of the present paper.

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[13] Since the perfect permutational symmetry is not required, the present result will be extended to a system with weak local couplings besides the global one. It cannot be directly applied to a system with local coupling only (such as coupled map lattice or partial differential equation). However, by considering the number of degrees of freedom that are directly coupled within a correlation length in space, the present argument may be applied to such a system.

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